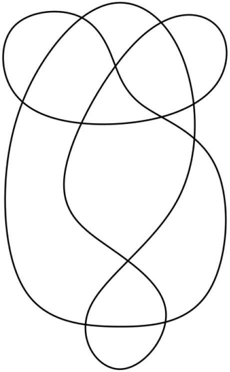
Conveniences of Alternating Diagrams

Ana Wright

October 2, 2019

What can be alternating?

There exists a choice of crossing information for any knot or link projection to be alternating.



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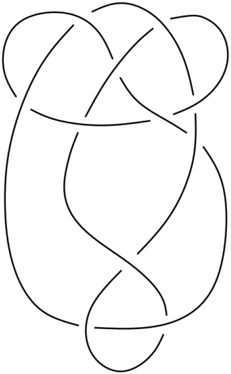

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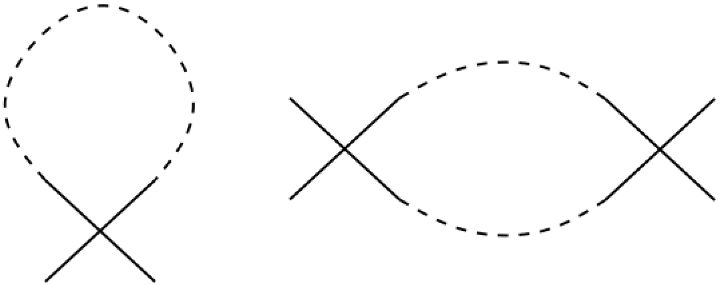

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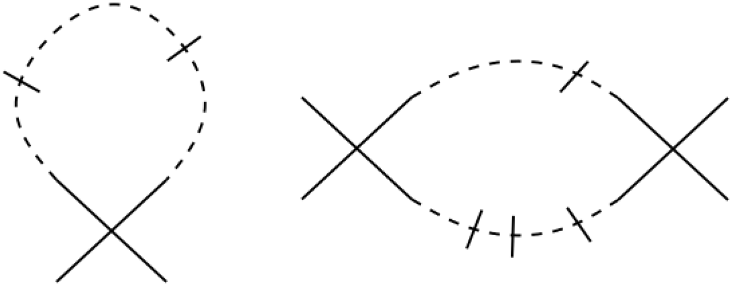
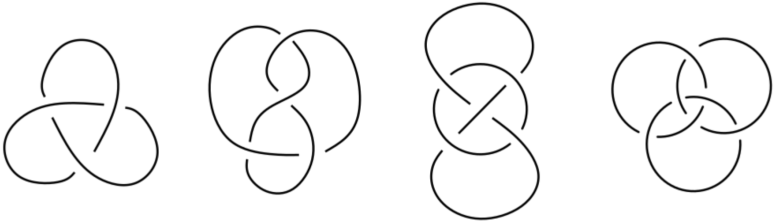


A lot of small links are alternating



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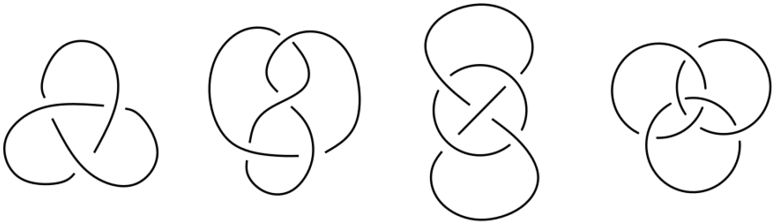
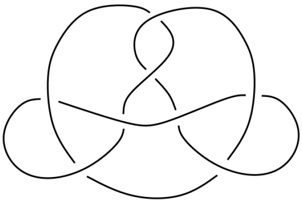
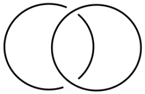


There exist non-alternating knots and links:



Results about alternating diagrams

Theorem (Menasco, 1984): Given an alternating diagram D of a link L, L is split if and only if D is split.



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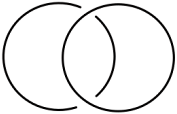

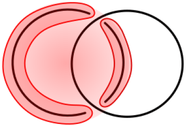

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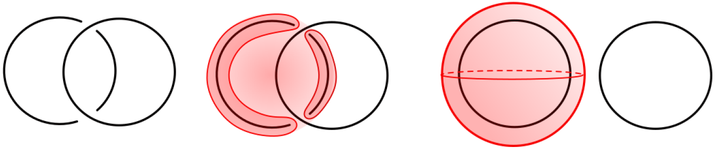

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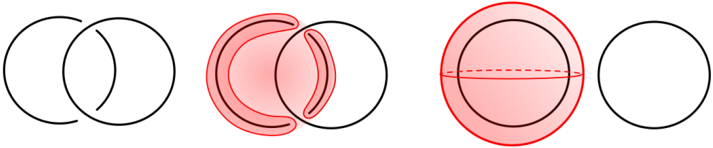

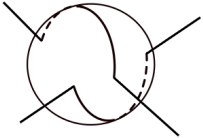

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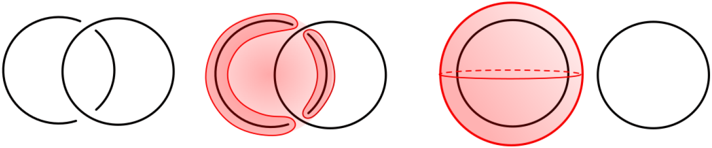
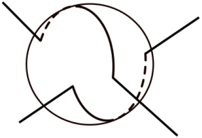
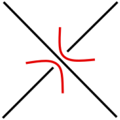
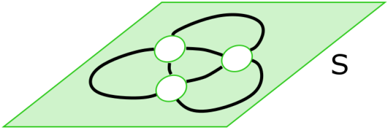






Set up for proof



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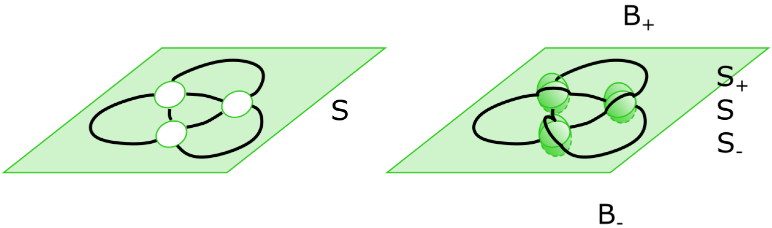

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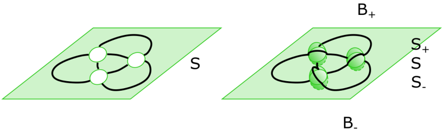
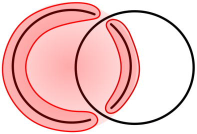


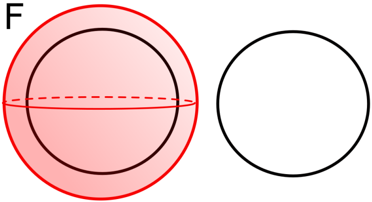
Standard Position for a surface F:

- $F \cap B_+$ and $F \cap B_-$ are each a disjoint union of discs
- No component of $F \cap S_+$ or $F \cap S_-$ meets any bubble in more than one arc.
- Each component of $F \cap S_+$ and $F \cap S_-$ goes through a crossing region.



Proof Sketch

Let L be a split link with an alternating diagram D. Assume by way of contradiction that D is not a split diagram. There is a sphere F splitting L.



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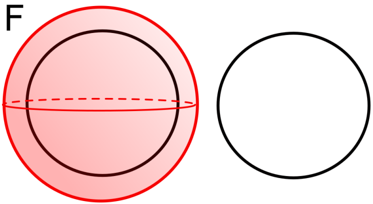

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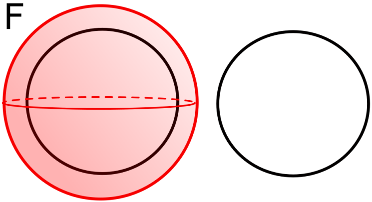

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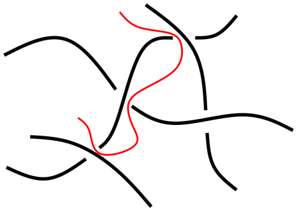

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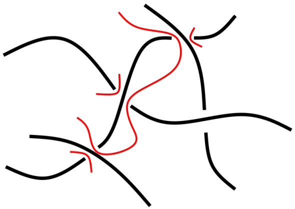

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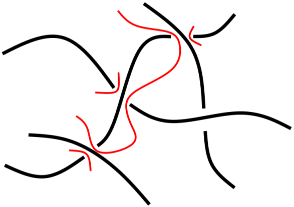

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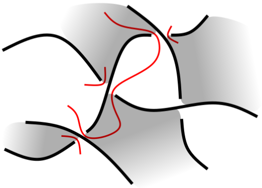
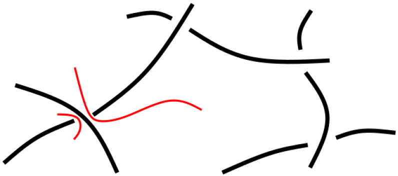


Let C be an innermost component of $F \cap S_+$. Since F is in standard position, C goes through a crossing region. Notice from the checkerboard coloring that C goes through an even number of crossing regions.

Proof Sketch

Since D is alternating, when C goes through a second crossing region, there will be a component of $F \cap S_+$ on the other side of C.



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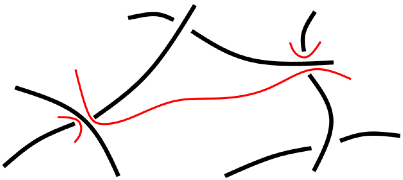

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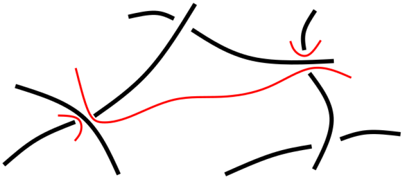
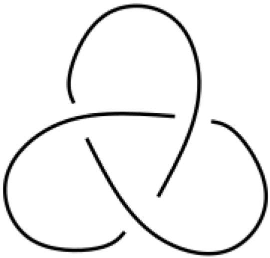


Therefore, C is not innermost. $\Rightarrow\Leftarrow$

Short Detour: Prime Knots



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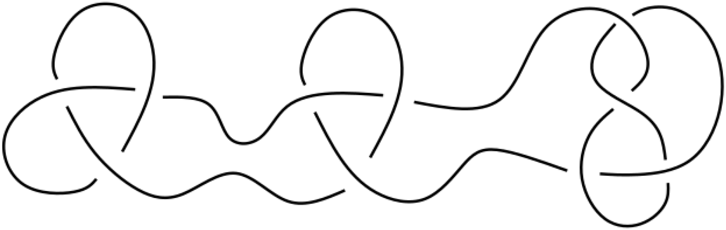

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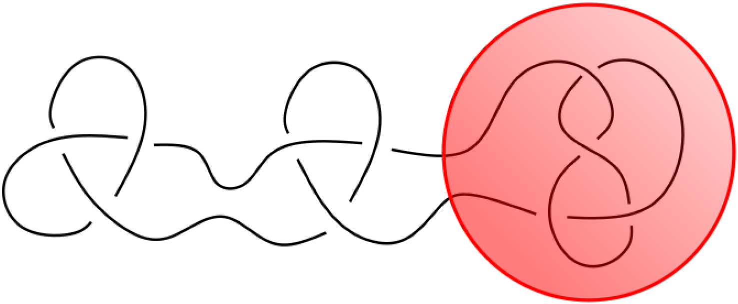

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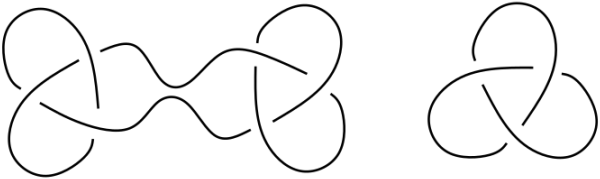

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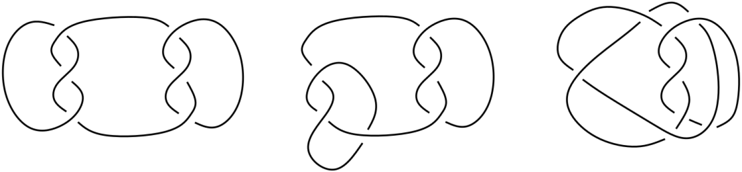

Results about alternating diagrams

Tait Conjectures

- Theorem (Kauffman, Murasugi, Thistlethwaite, 1987): Any reduced diagram of an alternating link has the fewest possible crossings.
- Theorem (Kauffman, Thistlethwaite, 1987): An amphichiral (or achiral) alternating link has zero writhe.
- Theorem (Thistlethwaite, Menasco, 1991): Given any two reduced alternating diagrams D_1 and D_2 of an oriented, prime alternating link: D_1 may be transformed to D_2 by means of a sequence of moves called flypes.

Results about alternating diagrams

Theorem (Crowell, Murasugi, 1958): Applying the Seifert algorithm to an alternating diagram results in a Seifert surface of minimal genus.

Theorem (Kauffman, Murasugi, Thistlethwaite, 1987): Every minimal crossing diagram of a prime alternating link is alternating.

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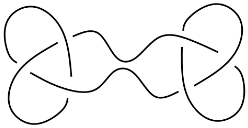
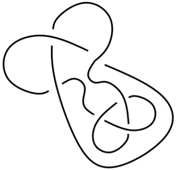
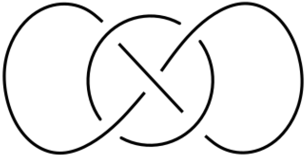
Examples

A non-alternating diagram of an alternating knot with minimal crossings.





Proof that the Whitehead link is not split.



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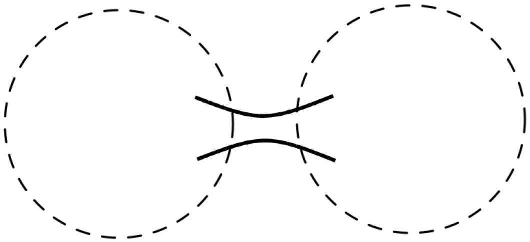

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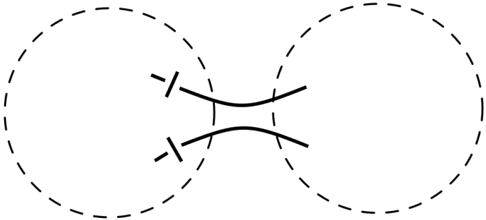

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